Model Theory - Lecture 5 - Quantifier Elimination "2"

We will prove

Theorem Let $\mathcal{T}$ be a theory (in FOL) If the set of finite partial isomorphisms has the B&F, then the theory eliminates quantifiers

Remark This result is "semantic" and we want a "syntactic" result and this will need completeness (implicitly) to link the two.

Corollary The theory of dense linear orders without end points eliminates quantifiers.

Exercise: Eliminate quantifiers from $\exists x((x>a) \wedge (x<b))$

The theory of infinite sets clearly has the B&F

Definition Let Γ be a set of formulas We write

$$\varphi \models \bigvee_{\gamma \in \Gamma} \gamma$$

if and only if $Mod(\varphi) \subseteq \bigcup_{\gamma \in \Gamma} Mod(\gamma)$

Theorem If $\varphi \models \bigvee_{\gamma \in \Gamma} \gamma$, then there exists a finite $\Gamma_0 \subseteq \Gamma$ such that

$$\varphi \models \bigvee_{\gamma \in \Gamma_0} \gamma$$

Proof Assume Γ_0 does not exist Then, for every finite $\Gamma' \subseteq \Gamma$, we have a model $M_{\Gamma'} \models \bigwedge_{\gamma \in \Gamma'} \neg\gamma$ Therefore, by Compactness, the theory $\varphi \cup \{\neg\gamma\}_{\gamma \in \Gamma}$ has a model, call it N

Then, $N \not\models \bigvee_{\gamma \in \Gamma} \gamma$ This is a contradiction, then Γ_0 exists

$\blacksquare$

Theorem (Separation) Let Γ be a set of formulas closed under $\wedge, \vee$, and φ_1, φ_2 be two (different) theories. Assume that, for every $\mathcal{M}_1 \in Mod(\varphi_1)$ and every $\mathcal{M}_2 \in Mod(\varphi_2)$, there exists $\gamma_{1,2} \in \Gamma$ such that $\mathcal{M}_1 \models \gamma_{1,2}$ and $\mathcal{M}_2 \models \neg\gamma_{1,2}$.

Then, there exists γ^* in Γ such that $\varphi_1 \models \gamma$ and $\varphi_2 \models \neg\gamma$

Proof Choose a model $A \in Mod(\varphi_1)$. For every $B \in Mod(\varphi_2)$,

we have $\varphi_2 \models \bigvee_{B \in Mod(\varphi_2)} \neg\gamma_{A,B}$ and so, by the previous theorem,

there exists a finite subset of $Mod(\varphi_2)$, say $\mathcal{B}$, such that

$$\varphi_2 \models \bigvee_{B \in \mathcal{B}} \neg\gamma_{A,B} \qquad (1)$$

Fix $B' \in \mathcal{B}$, we have

$$\varphi_1 \models \bigvee_{A \in Mod(\varphi_1)} \gamma_{A,B'},$$

and so, again, there exists a finite $\mathcal{A} \subseteq Mod(\varphi_1)$ such that

$$\varphi_1 \models \bigvee_{A \in \mathcal{A}} \gamma_{A,B'}$$

Now, for all $A \in \mathcal{A}$, consider $\mathcal{B}_A$. We take the formula constructed by conjuncting all of the disjunctions in (1) on different $\mathcal{B}_A$'s

$\blacksquare$

Remark: One can do with free variables in the formulas extending $\mathcal{L} \cup \{x_1, \ldots, x_n\}$ in the statement

Let φ be a theory and let $\Gamma = \Gamma'(x_1, \dots, x_n)$ in $\mathcal{L} \cup \{x_1, \dots, x_n\}$ be closed under $\wedge$ and $\vee$. Suppose that, given $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(\varphi)$ and given a choice of interpretation of the constants $\{x_1, \dots, x_n\}$.

Let $A := \{x_1^{\mathcal{M}}, \dots, x_n^{\mathcal{M}}\}$ and $B := \{x_1^{\mathcal{N}}, \dots, x_n^{\mathcal{N}}\}$, we have that $A \equiv_\Gamma B$. Then,

<u>Theorem</u> for every formula φ, then there exists a $\gamma \in \Gamma$ such that

$$\varphi \models \forall y \, (\varphi(y) \longleftrightarrow \gamma(y))$$

<u>Proof</u> Choose Γ to be the set of quantifier free formulas. By the previous theorem, we have that $\varphi \models \varphi(x_1, \dots, x_n) \longleftrightarrow$

$\longleftrightarrow \gamma(x_1, \dots, x_n)$ for every possible choice of the interpretation

"not" free variables, we are looking at their interpretation in the model

<u>Corollary</u>: if I is the family of finite partial isomorphisms and I has the B&F, then φ admits quantifier elimination.